Mathematical Analysis

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Chapter 3: Differential Calculus & Series Expansion

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Class Note Synopsis (WHOLE Part 1, 2, 3, 4)
B.Sc Semester 3
Subtopic: Function of a Single Variable, Cauchy's Criterion for finite limit, Continuity, Sequential Approach to Continuity, Continuity on interval, Uniform Continuity

...

Inverse function (Introduction), H.W, Limit f a Function (introduction)
Appendix:

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♦ *This is Draft overview version of classnote* ♦ *November 25, 2021*

Instruction/ Suggestion:

This is Draft overview version of classnote. Such draft overview version of classnotes will be given time-to-time as a draft synopsis of the class discussions. Dear students, you should make your own "handwritten" classnote for your own future references & you are advised to write down in details in your own notebook and complete all home works (H.W) that will be given time-to-time (refer to class discussions for solutions and hints).

PART (1)

1 Real valued function

Definition 1.1. *Let A, B be two non-empty subsets of $\Re$. f is a rule through which each element of A is associated with* **unique** *element of B. Then f is called* **mapping** *from A to B.*

$$f : A \to B \quad \text{we write } y = f(x), x \in A, y \in B.$$

Here $A =$ domain set, $B =$ range set.

Range, codomain

range /image.
range is a subset of codomain.
**** Refer to class discussion.

Example 1.1.

Indicator fn. $f(x) = I_A(x), x \in \Re,$ *where*

$$I_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}$$

Dirichlet fn. $f(x) = I_Q(x), x \in \Re,$ *where*

$$I_Q(x) = \begin{cases} 1 & if \quad x \in Q \\ 0 & if \quad x \in Q^c \end{cases}$$

Signum fn. $f(x) = sgn(x), x \in \Re,$ *where*

$$I_A(x) = \begin{cases} \frac{|x|}{x} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

Identity fn. $f(x) = x \quad x \in \Re$

Modulus fn. $f(x) = |x| \quad x \in \Re$

Box fn. $f(x) = [x] \quad x \in \Re$ (*write down proper definition of box function by yourself! This was done in class 11*)

Step fn. $f(x) = \sum_{k=1}^{n} c_k I_{A_k}(x), \quad x \in \Re,$ A_k *are piecewise disjoint intervals and* $\cup_k A_k = \Re$.

Even fn. $f(x), x \in \Re$ *is said to be even function if* $f(-x) = f(x)$.

Odd fn. $f(x), x \in \Re$ *is said to be odd function if* $f(-x) = -f(x)$.

Monotone increasing function. $f(x), x \in \Re$ *is said to be monotone increasing function if* $x_1 < x_2 \implies f(x_1) \leq f(x_2)$.

Monotone increasing function. $f(x), x \in \Re$ *is said to be monotone decreasing function if* $x_1 < x_2 \implies f(x_1) \geq f(x_2)$.

Hyperbolic function.

Hyperbolic functions occur in the calculations of angles and distances in hyperbolic geometry. In mathematics, hyperbolic functions are analogues of the ordinary trigonometric functions, but defined using the hyperbola rather than the circle.
The hyperbolic cosine is the function:

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

and the hyperbolic sine is the function

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Just as the points $(\cos t, \sin t)$ *form a circle with a unit radius, the points* $(\cosh t, \sinh t)$ *form the right half of the unit hyperbola.*
Also, just as the derivatives of $\sin(t)$ *and* $\cos(t)$ *are* $\cos(t)$ *and* $-\sin(t)$*, the derivatives of* $\sinh(t)$ *and* $\cosh(t)$ *are* $\cosh(t)$ *and* $+\sinh(t)$*.*

Animation: *https://en.wikipedia.org/wiki/File:HyperbolicAnimation.gif*, *https://en.wikipedia.org/wiki/Hyperbolic_functions*

2 Injection, Surjection, Bijection

Injections, surjections, and bijections are classes of functions distinguished by the manner in which arguments (from the domain) and images (output expressions from the codomain) are related or mapped to each other.

2.1 Injective Function

Definition 2.1. *The function is injective, or one-to-one, if each element of the codomain is mapped to by at most one element of the domain, or equivalently, if distinct elements of the domain map to distinct elements in the codomain. An injective function is also called an injection. Notationally:*

$$\forall x_1, x_2 \in A: \; x_1 \neq x_2 \; \Rightarrow f(x_1) \neq f(x_2).$$

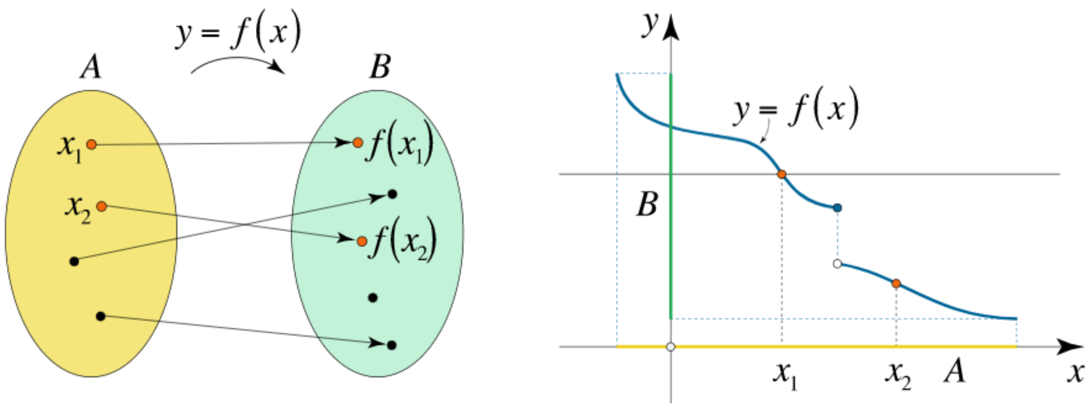

Figure 1: Injective function

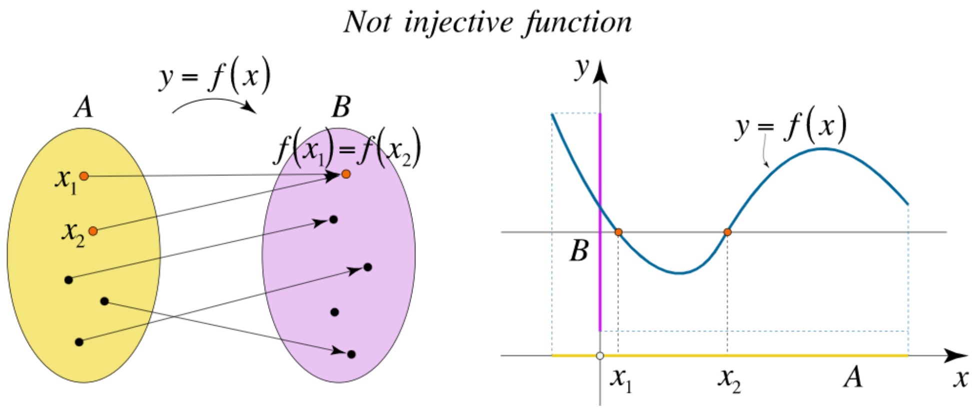

Figure 2: Not injective function

Horizontal Line test for injection

A horizontal line intersects the graph of an injective function at most once (that is, once or not at all). In this case, we say that the function passes the horizontal line test. If a horizontal line intersects the graph of a function in more than one point, the function fails the horizontal line test and is not injective.
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If a horizontal line intersects the graph of a function in more than one point, the function fails the horizontal line test and is not injective.

2.2 Surjection/ Surjective function/ Onto function

Definition 2.2. *A function f from A to B is called surjective (or onto) if for every y in the codomain B there exists at least one x in the domain A such that:*

$$\forall y \in B: \ \exists x \in A \text{ such that } y = f(x).$$

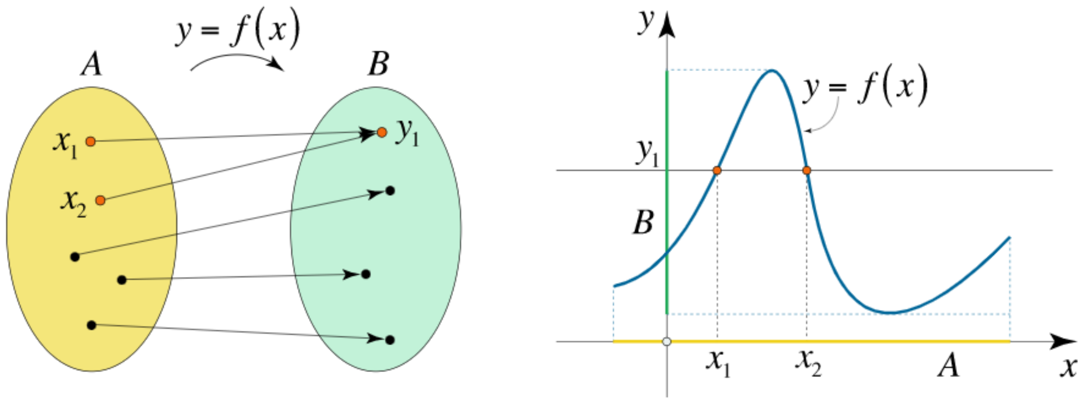

Figure 3: Surjective function

Horizontal Line test for surjection

The range and the codomain for a surjective function are identical.
Any horizontal line should intersect the graph of a surjective function at least once (once or more).

2.3 Bijective function /One-to-one correspondence

Definition 2.3. *A function f from A to B is called bijection (one-one + onto / one to one correspondence) if for every y in the codomain B there exists* **EXACTLY** *one x in the domain A such that:*

$$\forall y \in B : \exists! x \in A \text{ such that } y = f(x).$$

Horizontal Line test for Bijection

The range and the codomain for a surjective function are identical.

Any horizontal line passing through any element of the range should intersect the graph of a bijective function exactly once.

3 Appendix: More about Hyperbolic Functions

Definition 3.1. *The hyperbolic cosine is the function:*

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

and the hyperbolic sine is the function

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

. *The other hyperbolic functions are:*

$$\begin{aligned}
\tanh x &= \frac{\sinh x}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x} \\
sechx &= \frac{1}{\cosh x} \\
cschx &= \frac{1}{\sinh x}
\end{aligned}$$

Theorem 1. *The range of* $\cosh x$ *is* $[1, \infty)$

Proof. Let $y = \cosh x$. We solve for x:

$$\begin{aligned}
y &= \frac{e^x + e^{-x}}{2} \\
2y &= e^x + e^{-x} \\
2ye^x &= e^{2x} + 1 \\
0 &= e^{2x} - 2ye^x + 1 \\
e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\
e^x &= y \pm \sqrt{y^2 - 1}
\end{aligned}$$

From the last equation, we see $y^2 \geq 1$, and since $y \geq 0$, it follows that $y \geq 1$.
Now suppose $y \geq 1$. So, $y \pm \sqrt{y^2 - 1} > 0$. Then $x = \ln(y \pm \sqrt{y^2 - 1})$ is a real number and $y = \cosh x$. Hence, y is in the range of $\cosh(x)$. □

H. W.

(1) Show that, The domain of $\coth x$ and $cschx$ is $x \neq 0$ while the domain of the other hyperbolic functions is all real numbers.
(2)show that if $y = \sinh x$, then $x = \ln(y + \sqrt{y^2 + 1})$. Using this (or may not), Show that the range of $\sinh x$ is all real numbers.
(3) Show that the range of $tanhx$ is $(-1, 1)$. What are the ranges of $cothx, sechx, andcschx$? (Hint: Use the fact that they are reciprocal functions.)

Theorem 2. *For all* $x \in \Re$, $\cosh^2 x - \sinh^2 x = 1$

Proof.

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1.$$

□

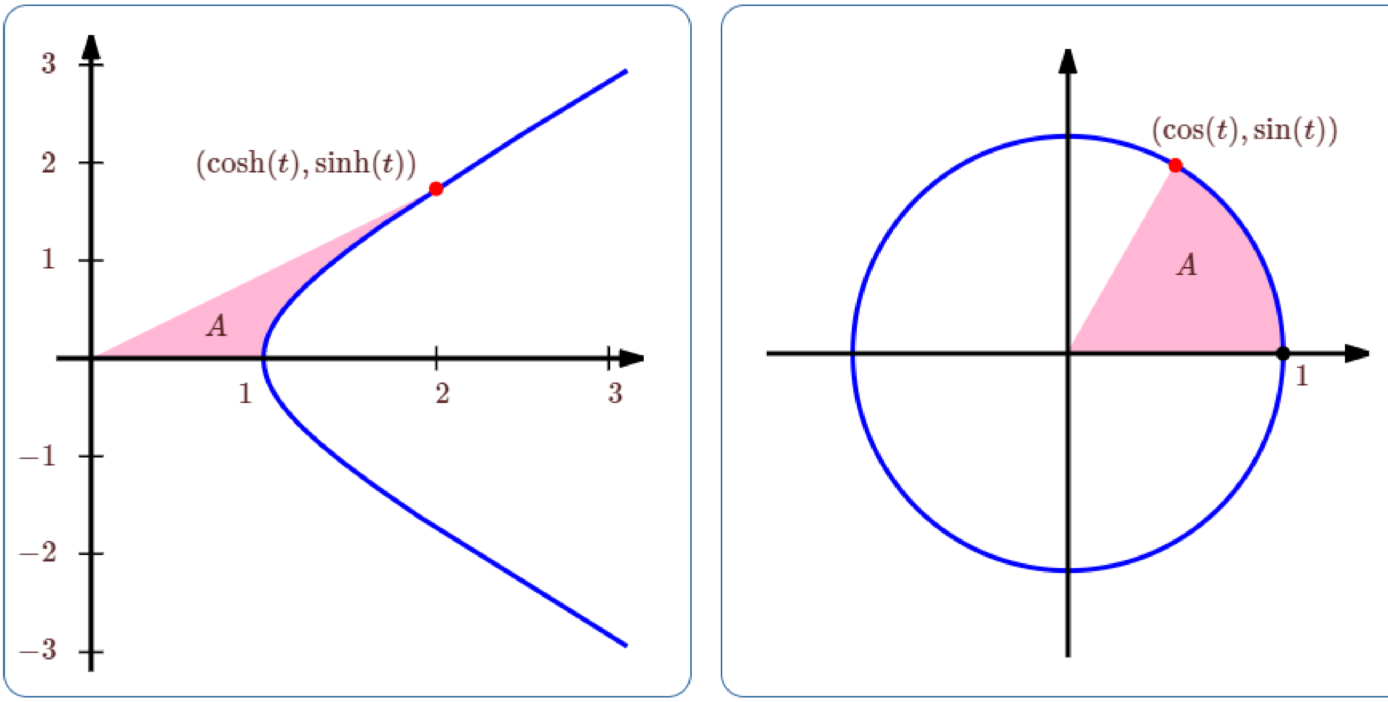

Figure 4: Geometric definitions of $\sin, \cos, \sinh, \cosh$: Here t is twice the shaded area in each figure. (Prove that, (H.W)

H.W

Show the following in a similar manner:

$$1 - tanh^2 x = sech^2 x \qquad \text{and} \qquad coth^2 x - 1 = csch^2 x.$$

4 References

1. Apostol: Mathematical Analysis
2. Wekepedia reference. https://en.wikipedia.org/wiki/Bijection,_injection_and_surjection
3. https://math24.net/injection-surjection-bijection.html

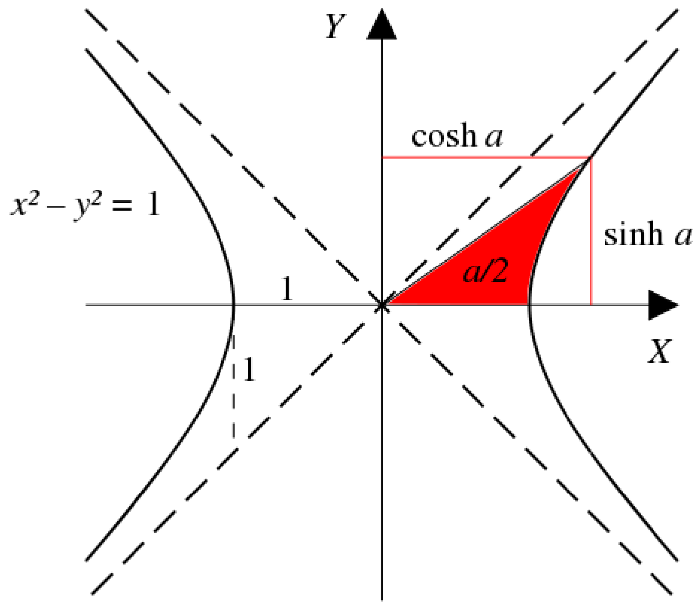

Figure 5: A ray through the unit hyperbola $x^2 - y^2 = 1$ at the point $(\cosh a, \sinh a)$, where a is twice the area between the ray, the hyperbola, and the x-axis. For points on the hyperbola below the x-axis, the area is considered negative. Reference: https://www.whitman.edu/mathematics/calculus_online/section04.11.html

PART 2

5 H.W. from previous class

Home work 1. 1. *Show that $f : \Re \to \Re$ defined by $f(x) = x^3$ is bijective function.*

***T/F Check whether $f : \Re^+ \to \Re^+$ defined by $f(x) = x^2$ is bijective function.*

***T/F Check whether $f(x) = [x]$ is bijective function.*

2. *Let $f : \Re \to \Re$ defined by $f(x) = x(x-1)(x-2)$. Then:*

   (a) *f is one-one but not onto.*
   (b) *f is not one-one but is onto.*
   (c) *f is neither one-one nor onto.*
   (d) *f is one-one and onto.*

3. *Check whether $f : \Re \to \Re$ defined by $f(x) = t(x)[t(x) + t(-x)]$ is ven function / odd function if*

   (a) *if f even function*
   (b) *if f odd function*

6 Inverse of Real valued function

an inverse function (or anti-function) is a function that "reverses" another function: if the function f applied to an input x gives an output y, then applying its inverse function g (denoted as f^{-1} to y gives the result x.

Definition 6.1 (Inverse of a function:)**.** *Let f be a function whose domain is the set x, and whose codomain is the set Y. Then f is invertible if there exists a function g with domain Y and codomain X, with the property:*

$$f(x) = y \iff g(y) = x$$

Theorem 3 (Uniqueness of Inverse)**.** *If f is invertible, then the function g is unique. which means that there is exactly one function g satisfying this property. Moreover, it also follows that the ranges of g and f equal their respective codomains (both bijection). The function g is called the inverse of f, and is usually denoted as f^{-1}.*

Example 6.1. *(1) To convert Fahrenheit to Celsius: $f(F) = (F - 32) \times 5/9$. The Inverse Function (Celsius back to Fahrenheit): $f^{-1}(C) = (C \times 9/5) + 32$*
*(2) $f(x) = x^2$ does not have an inverse but $\{x^2 | x \geq 0\}$ does have an inverse. (** Refer to class discussion)*

Home work 2.

1. *Show that, $(f^{-1})^{-1} = f$*
2. *If f^{-1} exists then show that f is bijective function.*
3. *The graph of $f(x)$ and $f^{-1}(x)$ are symmetric across the line $y = x$.*
4. *Find f^{-1} if $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$, where $|x| < 1$*

7 Limit of a function

7.1 Recall !

In calculus, you have already learnt limit. Intuitively,
Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number.
If all values of the function $f(x)$ approach the real number L as the values of $x (\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L.
(As x gets closer to a, $f(x)$ **gets closer and stays close** to L.) Symbolically, we express this idea as $\lim_{x \to a} f(x) = L$.

8 Analytical viewpoint of "Limit" of function

8.1 Previously (in 1 st Chapter) studied definition

Definition 8.1. *A function $f(x)$ is said to tend to a limit L as x tends to a (we write $\lim_{x \to a} f(x) = L$.) if :*

$$\forall \epsilon > 0, \ \exists \delta \ \ni |f(x) - L| < \epsilon \ \forall \ |x - a| < \delta$$

H.W.
Again Study & Write in notebook the followings: from Textbook (e.g., Apostol)

- $\epsilon - \delta$ Definition of limit of function when limit $L = +\infty$
- $\epsilon - \delta$ Definition of limit of function when limit $L = +\infty$
- $\epsilon - \delta$ Definition of limit of function for Right hand limit
- $\epsilon - \delta$ Definition of limit of function for Left hand limit
- Properties of limit (already studied in 11-12 syllabus)

Home work 3. *Using definition of limit, show that* $\lim_{x\to 0} x \sin \frac{1}{x} = 0$

H.W. hint: See any textbook for similar problem and its solution (e.g., Apostol). □

8.2 Sequential Definition of Limit of a function

We can combine some of the concepts that we have introduced before:

1. functions,
2. sequences,
3. topology

. If we have some function $f(x)$ and a given sequence a_n, then we can apply the function to each element of the sequence, resulting in a new sequence.
What we need is that if the original sequence converges to some number L, then the new sequence $f(a_n)$ should converge to $f(L)$, and if the original sequence diverges, the new one should diverge also.

Definition 8.2 (Limit of a Function (sequences version))**.** *A function f with domain $D \subset \Re$ converges to a limit L as x approaches a number c if $D - \{c\}$ is not empty and for* **any sequence** *$\{x_n\} \in D - \{c\}$ that converges to c, the sequence $\{f(x_n)\}$ converges to L. We write $\lim_{x\to c} f(x) = L$*

Usefulness of Sequential Definition of limit

Show that, $\lim_{x\to 0} \frac{|x|}{x}$ does not exists.
Proof. Let us consider two sequences $\{x_n\}, \{y_n\}$ where $x_n = 1/n \to 0$ and $y_n = -1/n \to 0, n \in \mathbb{N}$.
Then, $f(x_n) = 1 \to 1$ and $f(y_n) = -1 \to -1$ (trivially).
Therefore, $\lim_{x\to 0} \frac{|x|}{x}$ does not exists, as it is violating the sequential definition of limit. □

**** Refer to class discussion.

Home work 4. *Determine whether the following limits exists or not:*

1. $\lim_{x\to 0} \cos(1/x)$
2. $\lim_{x\to 0} [x]$

3. $\lim_{x\to 0} sgn(x)$

4. $\lim_{x\to 0} c1/x$

5. $\lim_{x\to 0} 1/x \cdot \sin(1/x)$

Proof. Hint for H.W Take suitably (** class discussion)
$x_n = \dfrac{1}{n+1}$, $y_n = \dfrac{-1}{n+1}$,
$x_n = \dfrac{2}{\pi(4n+1)}$, $y_n = \dfrac{1}{n\pi}$ □

Home work 5. *Show that,* $\lim_{x\to\infty} x\sin(x)$ *does not exists in* $\Re \cup \{+\infty\}$.

Home work 6 (Do this Experiment :)**.** *Consider the function f, where* $f(x) = 1$ *if* $x \leq 0$ *and* $f(x) = 2$ *if* $x > 0$*. The sequence* $\{1/n\}$ *converges to 0.*

1. *What happens to the sequence* $\{f(1/n)\}$ *?*

2. *The sequence* $\{3 + (-1)^n\}$ *is divergent. What happens to the sequence* $\{f(3 + (-1)^n)\}$ *?*

3. *The sequence* $\{(-1)^n/n\}$ *converges to* 0*. What happens to the sequence* $\{f((-1)^n/n)\}$ *?*

9 References

1. Apostol: Mathematical Analysis

2. Wekepedia references

PART (3)

10 Cauchy Criterion for Finite Limits

Theorem 4. *A real valued function f tends to a finite limit as $x \to c$ iff for every $\epsilon > 0$ $\exists$ a deleted neighbourhood $N'(c)$ of c such that*

$$|f(x_1) - f(x_2)| < \epsilon, \ \forall \ x_1, x_2 \in N'(c)$$

Recall: The Cauchy criterion is a characterization of convergent sequences of real number: Done in previous classes!

The Cauchy criterion is a characterization of convergent sequences of real numbers. It states that

Theorem 5. *A sequence $\{a_n\}$ of real numbers has a finite limit if and only if for every $\epsilon > 0$ there is an N such that*

$$|a_n - a_m| < \epsilon \quad \forall \ n, m \geq N.$$

This is called the Cauchy condition and a sequence which satisfies it is called Cauchy sequence.

10.1 Examples:

Example 10.1. *Let*

$$f(x) = \begin{cases} x \ if \ x \in \boldsymbol{Q} \\ -x \ if \ x \in \boldsymbol{Q}^c \end{cases}$$

Show that, $\lim_{x \to a} f(x)$ *exists only when* $a = 0$

Proof. If $a > 0$, let $x_1 \in \boldsymbol{Q}$ and $x_2 \in \boldsymbol{Q}^c$ in $(a, a+\delta)$
Hence, $|f(x_1) - f(x_2)| = |x_1 + x_2| > 2a \nless \epsilon$, so the limit does not exists.
If $a < 0$, let $x_1 \in \boldsymbol{Q}$ and $x_2 \in \boldsymbol{Q}^c$ in $(a, a+\delta)$
Hence, $|f(x_1) - f(x_2)| = |x_1 + x_2| > 2a \nless \epsilon$, so the limit does not exists.
If $a = 0$, $|f(x) - 0| = |x| < \delta(= \epsilon)$, so the limit exists only when $a = 0$ and the limit is 0. □

10.2 Home works

Example 10.2 (Home work:)**.** *Let*

$$f(x) = \begin{cases} 1 \ if \ x \in \boldsymbol{Q} \\ -1 \ if \ x \in \boldsymbol{Q}^c \end{cases}$$

Show that, $\lim_{x \to a} f(x)$ *does not exists.*

Example 10.3 (Home work:)**.** *Let*

$$\phi(x) = \lim_{n \to \infty} \frac{x^n f(x) + g(x)}{x^n + 1}$$

then show that

$$\phi(x) = \begin{cases} f(x)\ if\ |x| > 1 \\ g(x)\ if\ |x| < 1 \\ (f(x)+g(x))/2\ if\ x = 1 \\ undefined\ if\ x = -1 \end{cases}$$

Example 10.4 (Home work:). *Find the value of the limit:*

$$\lim_{x\to\infty} x\left(\exp(-x)\cos(4x) + \sin(1/4x)\right)$$

Hint:

$$\lim_{x\to\infty} x\left(\exp(-x)\cos(4x)\right) = 0, \quad \& \quad \lim_{x\to\infty} (\sin(1/4x)/(1/x) = 1/4)$$

□

11 Continuity at a point

Definition 11.1. *Continuity at a point: Let f be a function defined on an interval $[a, b]$. Then f is called continuous at c for $a < c < b$, if $\lim_{x\to c} f(x) = f(c)$.*

Definition 11.2. *Alternate definition: Let f be a function defined on an interval $[a, b]$. Then f is called continuous at c for $a < c < b$, if $\lim_{x\to c} f(x) = f(c)$, i.e.,*

$$\forall\ \epsilon > 0,\ \exists\delta > 0\ such that, |f(x) - f(c)| < \epsilon\ for\ 0 < |x - c| < \delta.$$

H.W.
Again Study & Write in notebook the followings: from Textbook (e.g., Apostol)

- Continuity of function from Left at c
- Continuity of function from right at c
- Continuity of function from both left & Left at c
- Continuity of function (both L & R) at c
- Removable Discontinuity
- Non-removable discontinuity
- Removable Discontinuity (First kind- Both L, R limit exists but not equal)
- Removable Discontinuity (First kind from Left - Both L, R limit exists but not equal, $L = f(c)$)
- Removable Discontinuity (First kind from right - Both L, R limit exists but not equal, $R = f(c)$)
- Removable Discontinuity (Second kind from Left - left limit does not exists)
- Removable Discontinuity (First kind- Both L, R limit exists but not equal)
- Properties of continuity (already studied in 11-12 syllabus)

Example 11.1 (Home work:). *Let*

$$f(x) = \begin{cases} x\sin\frac{1}{x} \ if \ x \neq 0 \\ 0 \ if \ x = 0 \end{cases}$$

Show that, $f(x)$ *is continuous when* $x = 0$

Example 11.2 (Home work:). *Let*

$$f(x) = \begin{cases} x\sin\frac{1}{x} \ if \ x \neq 0 \\ 1 \ if \ x = 0 \end{cases}$$

Show that, $f(x)$ *has a removable discontinuity at* $x = 0$

Example 11.3 (Home work:). *Show that,* $\sin 1/x$ *is discontinuious at* $x = 0$

Example 11.4 (Home work:). *Examine continuity at* $x = 0$

$$f(x) = \begin{cases} \dfrac{x\exp(1/x)}{1+\exp(1/x)} \ if \ x \neq 0 \\ 0 \ if \ x = 0 \end{cases}$$

12 Sequential Approach to Continuity

Informally, we have to use sequential definition of limit here.
To be continued $\cdots$

13 References

1. Apostol: Mathematical Analysis
2. Wekepedia references

PART 4,5

14 Continuity in an interval

Definition 14.1. *Continuity in an interval A function f is said to be continuous in an interval $[a, b]$ if it is continuous at every point of $[a, b]$.*

Example 14.1. *Show that, $f(x) = x$ is continuous on $\Re$. Let c be any arbitrary point on $\Re$.*

$$|f(x) - f(c)| = |x - c|$$

So, for any $\epsilon > 0$,

$$|f(x) - f(c)| < \epsilon \quad \text{if } 0 < |x - c| < \delta(= \epsilon)$$

Therefore, $f(x)$ is continuous at c for any $c \in \Re$.

Example 14.2. *Home work:*

1. *Show that, $f(x) = \sin x$ is continuous.*
2. *Show that, $f(x) = x^2$ is continuous.*

Proof. Hint:

$$\begin{aligned}
&|x + c||x - c| = |x - c||x - c + 2c| \leq |x - c|(|x - c| + |2c|).\\
&Now, \quad |x - c| < \delta < 1 \quad \text{(argue yourself!)}\\
&so, \quad |x^2 - c^2| \leq |x - c|(1 + 2|c|) < \epsilon.
\end{aligned}$$

□

Example 14.3. *Home work: Let $f(x), g(x)$ two function continuous at c. Show that:*

1. *$f(x) + g(x)$ also continuous at c.*
2. *$f(x) - g(x)$ also continuous at c.*
3. *$f(x) \cdot g(x)$ also continuous at c.*
4. *$f(x)/g(x)$ also continuous at c if $g(c) \neq 0$.*

Hint: Refer to Apostol.

Example 14.4. *Home work:*

1. *Show that, if $f(x)$ continuous then $|f(x)|$ is also continuous.*
2. *Is the converse true?*

15 Sequential Approach to Continuity

Definition 15.1. *Sequential Approach to Continuity:* $f : A \to \mathbb{R}$ *is continuous at a point* $a \in A$ *if for EVERY sequence* (x_n) *in* A *such that* $x_n \to a$*, we have* $f(x_n) \to f(a)$*.*

Definition 15.2. *Alternate Definition: For every sequence* $\{c_n\}$ *in* $[a, b]$*,*

$$\lim_{n\to\infty} c_n = c \implies \lim_{n\to\infty} f(c_n) = f(c).$$

Each sequence means that, no matter what sequence $(x_n)_{n\geq 1}$ we pick, if $x_n \to a$, then $f(x_n) \to f(a)$.

Extra Knowledge:

proof of: the limit definition is equivalent to the sequential definition:
**** See Apostol. In class, Will be done later.

Topologist's sine curve is discontinuous at 0:

Topologist's sine curve:

$$T = \left\{\left(x, \sin\tfrac{1}{x}\right) : x \in (0, 1]\right\} \cup \{(0, 0)\}.$$

Topologist's sine curve is discontinuous at 0. The reason is that we can find two different sequences which both converge to 0, but where the limits of sequences $f(x_i)$ are different.

1. take for the first sequence

$$\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \frac{1}{4\pi}, \dots, \frac{1}{n\pi}, \cdots$$

   This sequence certainly converges to 0. Since the function is $f(x) = \sin(1/x)$, the corresponding $f(x_i)$ sequence is $\sin(1/(1/n\pi)) = \sin(n\pi) = 0$. In other words, this sequence is $0, 0, 0, \cdots$. The limit of this sequence is zero.

2. Alternatively, consider the sequence

$$\frac{1}{\pi/2}, \frac{1}{2\pi + \pi/2}, \frac{1}{4\pi + \pi 2}, \dots, \frac{1}{2n\pi + \pi/2}, \cdots$$

   Again, this sequence also converges to 0. But this time when we evaluate the function at these points, we get $\sin(2n\pi + \pi/2) = \sin(\pi/2) = 1$. So the sequence is $1, 1, 1, 1, \cdots$ and so the limit is 1 (and not 0).
   Hence,

Theorem 6. *If* $f : X \to Y$ *and* $g : Y \to Z$ *are continuous, then so is the composition* $g \circ f : X \to Z$*.*

Proof. Refer to Apostol. □

Example 15.1. *Home Works* **** *Will be updated later!*

16 Uniform continuity

Definition 16.1. *If X and Y are subsets of the real line,then the definition for uniform continuity of f: for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all such that for all*

$$x, y \in X, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

The difference between uniform continuity, versus ordinary continuity at every point, is that in uniform continuity the value of δ depends only on ε and not on the point in the domain.

Example 16.1. *Any constant function $f : D \to \Re$, is uniformly continuous on its domain. Solution: given $\epsilon > 0$, $|f(u) - f(v)| = 0 < \epsilon$ for all $u, v \in D$ regardless of the choice of δ.*

Theorem 7. *If $f : D \to \Re$ is uniformly continuous on D , then f is continuous at every point $x_0 \in D$.*

uniform continuity is a property of a function on a set, whereas continuity is defined for a function in a single point.

Continuous functions can fail to be uniformly continuous if they are unbounded on a bounded domain, such as $f(x) = \frac{1}{x}$ on $(0, 1)$, or if their slopes become unbounded on an infinite domain, such as $f(x) = x^2$ on the real line.

Example 16.2. *Let $f : \Re \to \Re$ be given by $f(x) = 7x - 2$. Show that f is uniformly continuous on $\Re$. Answer: Let $\epsilon > 0$ and choose $\delta = \epsilon/7$. Then, if $u, v \in \Re$ and $|u - v| < \delta$, we have*

$$|f(u) - f(v)| = |7u - 2 - (7v - 2)| = |7(u - v)| = 7|u - v| < 7\delta = \varepsilon$$

Example 16.3. *Let $f : \Re \to \Re$ be given by $f(x) = \dfrac{x^2}{x^2 + 1}$. Show that f is uniformly continuous on $\Re$. Solution: Let $\epsilon > 0$.*

$$\begin{aligned}
\left| \frac{u^2}{u^2+1} - \frac{v^2}{v^2+1} \right| &= \left| \frac{u^2\left(v^2+1\right) - v^2\left(u^2+1\right)}{\left(u^2+1\right)\left(v^2+1\right)} \right| \\
&= \frac{|u-v||u+v|}{\left(u^2+1\right)\left(v^2+1\right)} \leq \frac{|u-v|(|u|+|v|)}{\left(u^2+1\right)\left(v^2+1\right)} \\
&\leq \frac{|u-v|\left(\left(u^2+1\right)+\left(v^2+1\right)\right)}{\left(u^2+1\right)\left(v^2+1\right)} \\
&\leq |u-v|\left(\frac{1}{v^2+1} + \frac{1}{u^2+1}\right) \leq 2|u-v|,
\end{aligned}$$

where we used that $|x| \leq x^2 + 1$ for all $x \in \Re$.
Now take $\delta = \epsilon/2$. Given $u, v \in \Re$ satisfying $|u - v| < \delta$ we have

$$|f(u) - f(v)| = \left| \frac{u^2}{u^2+1} - \frac{v^2}{v^2+1} \right| \leq 2|u - v| < 2\delta = \varepsilon.$$

Example 16.4. *Let* $f : (0,1) \to \mathbb{R}$ *be given by* $f(x) = \frac{1}{x}$*., Continuous but not uniformly continuous on* $(0,\infty)$ *. Solution: This function is continuous at every* $x \in (0,1)$ *(Show the continuity, Home work).*
We will show that f is not uniformly continuous on (0,1).
Let $\varepsilon = 2$ *and* $\delta > 0$ *. Take* $\delta_0 = \min\{\delta/2, 1/4\}$ *,* $x = \delta_0, y = 2\delta_0$*. Then* $x, y \in (0,1)$ *and* $|x-y| = \delta_0 < \delta$ *, but*

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \left|\frac{\delta_0}{2\delta_0^2}\right| = \left|\frac{1}{2\delta_0}\right| \geq 2 = \varepsilon.$$

This shows f is not uniformly continuous on (0,1) .

Roughly speaking,

f is uniformly continuous if, roughly speaking, it is possible to guarantee that $f(x)$ and $f(y)$ be as close to each other as we please by requiring only that x and y be sufficiently close to each other; unlike ordinary continuity, where the maximum distance between $f(x)$ and $f(y)$ may depend on x and y themselves.

To be continued $\cdots$

Theorem 8. *Sequential Criteria for Uniform Continuity: Let D be a nonempty subset of* $\Re$ *and* $f : D \to \mathbb{R}$ *. Then f is uniformly continuous on D if and only if the following condition holds:*
for every two sequences $\{u_n\}$*,* $\{v_n\}$ *in D such that* $\lim_{n\to\infty}(u_n - v_n) = 0$*, it follows that* $\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0$

Using Sequential criteria, easier proof of :
Let $f : (0,1) \to \mathbb{R}$ **be given by** $f(x) = \frac{1}{x}$**.,**
Continuous but not uniformly continuous on $(0,\infty)$

Proof. Consider the two sequences $u_n = 1/(n+1)$ and $v_n = 1/n$ for all $n \geq 2$. Then $\lim_{n\to\infty}(u_n - v_n) = 0$, but

$$\lim_{n\to\infty}(f(u_n) - f(v_n)) = \lim_{n\to\infty}\left(\frac{1}{1/(n+1)} - \frac{1}{1/n}\right) = \lim_{n\to\infty}(n+1-n) = 1 \neq 0.$$

□

16.1 continuity implies uniform continuity

Let $f : D \to \mathbb{R}$ be a continuous function. Suppose D is compact. Then f is uniformly continuous on D.

Proof. Refer to Apostol □

Theorem 9. *Let* $a, b \in \mathbb{R}, a < b$*. A function* $f : (a,b) \to \mathbb{R}$ *is uniformly continuous if and only if f can be extended to a continuous function* $\tilde{f} : [a,b] \to \mathbb{R}$ *. In other words, if there is a continuous function* $\tilde{f} : [a,b] \to \mathbb{R}$ *such that* $f = \tilde{f}_{|(a,b)}$*).*

Proof. Refer to Apostol ☐

Home Works:

Example 16.5. *Prove that each of the following functions is not uniformly continuous on the given domain:.*

1. $f(x) = x^2$
2. $f(x) = \sin \frac{1}{x}$ *on* $(0, 1)$
3. $f(x) = \log(x)$ *on* $(0, \infty)$

Example 16.6. *Prove that each of the following functions is uniformly continuous on the given domain:*

1. $f(x) = 1/x$ *on* $[a, \infty), \quad a > 0$
2. $f(x) = ax + b, a, b \in \mathbb{R}$

Example 16.7. *Let* $a, b \in \Re$ *and let* $f : (a, b) \to \mathbb{R}$.
Prove that if f *is uniformly continuous, then* f *is bounded. Prove that if* f *is continuous, bounded, and monotone, then it is uniformly continuous.*

17 References

1. Apostol: Mathematical Analysis
2. Wekepedia references
3. https://math.libretexts.org/Bookshelves/Analysis/Introduction_to_Mathematical_Analysis_I_(Lafferriere_Lafferriere_and_Nguyen)/03%3A_Limits_and_Continuity/3.05%3A_Uniform_Continuity